The Sum and Chain Rules for Maximal Monotone Operators

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Abstract. This paper is primarily concerned with the problem of maximality for the sum $A+B$ and composition L^*ML in non-reflexive Banach space settings under qualifications constraints involving the domains of A, B, M. Here X, Y are Banach spaces with duals X^*, Y^*, $A, B : X \rightrightarrows X^*$, $M : Y \rightrightarrows Y^*$ are multi-valued maximal monotone operators, and $L : X \to Y$ is linear bounded. Based on the Fitzpatrick function, new characterizations for the maximality of an operator as well as simpler proofs, improvements of previously known results, and several new results on the topic are presented.

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1. Introduction

The problem concerning the maximality of the sum of two maximal monotone operators was first stated and solved by Rockafellar in reflexive Banach spaces followed by a sum rule for the convex subdifferential in general Banach spaces (see [8, Theorem 1], [15, Theorem 2.8.3]). At the same time the conjecture which states that the reflexivity of the space can be avoided was formulated. Later, the sum rule for full-space domain operators was proved by Heisler (see [10, Theorem 37.4] or Theorem 4.1 below) and for single-valued linear operators by Phelps and Simons (see [6, Theorem 7.2]). Recently, Voisei [11,12,13] proved similar calculus rules for closed convex domain monotone operators in non-reflexive Banach spaces under weaker forms of the qualification constraint and solved completely the linear case (see [12,13] or Corollary 5.14. below). Using a topological argument, the sum rule for operators with the intersection of their domain interiors non-empty was shown to hold by Borwein (see [1]). Chain rules in the context of reflexive Banach spaces were obtained by Penot [5], Zălinescu [14], and Borwein [1].

In the present note we want to shed a new light on the ideas of the proofs and present new points of view as well as simpler arguments, improvements of some of the past results, and new results concerning the maximality of the sum or of the precomposition with a linear operator in the non-reflexive Banach space setting.

The plan of the paper is as follows. Next section introduces the Fitzpatrick and Penot functions together with their main features. In Section 3 new characterizations for the maximality and representability of an operator are discussed. Section 4 contains a simple proof of Heisler's result. Section 5 deals with the calculus of maximal monotone operators. This paper concludes with some improvements of the results contained in Voisei [12,13] and several other new results on the topic.

2. The natural dual system. The Fitzpatrick and Penot functions

Let X be a Banach space with dual X^* and bi-dual X^{**}. A multi-valued operator $A : D(A) \subset X \rightrightarrows X^*$ is called *monotone* if, for every $x_1, x_2 \in D(A)$, $x_1^* \in Ax_1$, $x_2^* \in Ax_2$

$$\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0, \tag{1}$$

where

$$p(x, x^*) = \langle x, x^* \rangle = x^*(x),\ (x, x^*) \in X \times X^*,$$

stands for the duality pairing in $X \times X^*$.

For the sake of notation simplicity we identify operators with their graphs and write

$$x \in D(A),\ x^* \in Ax \Leftrightarrow (x, x^*) \in A. \tag{2}$$

With this notation A is monotone iff $p(a_1 - a_2) \geq 0$ for every $a_1, a_2 \in A$.

A monotone operator is considered *maximal monotone* if it is maximal in the sense of inclusion in $X \times X^*$.

Let $Z = X \times X^*$. The natural dual system is formed by (Z, Z) with the dual product

$$(x, x^*) \cdot (y, y^*) = x^*(y) + y^*(x),\ \ x, y \in X,\ \ x^*, y^* \in X^*. \tag{3}$$

The convex conjugate with respect to the natural duality of $f : Z \to \mathbb{R} \cup \{\infty\}$ is given by

$$f^*(w) = \sup\{w \cdot z - f(z);\ z \in Z\},\ \ w \in Z, \tag{4}$$

Notice that

$$z \cdot z = 2p(z),\ \ p(\lambda z) = \lambda^2 p(z), \tag{5}$$

$$p(z_1 \pm z_2) = p(z_1) + p(z_2) \pm z_1 \cdot z_2 \tag{6}$$

$$p(z_1 + z_2) + p(z_1 - z_2) = 2(p(z_1) + p(z_2)), \tag{7}$$

for every $\lambda \in \mathbb{R}$, $z, z_1, z_2 \in Z$.

On Z we fix a topology compatible with the natural duality such as the strong$\times$weakly-star topology. In the sequel all topological notions will be understood with respect to this fixed topology on Z if not otherwise specified.

For $\emptyset \neq A \subset Z$ let

$$p_A = p + i_A, \tag{8}$$

where $i_A(z) = 0$, if $z \in A$, $i_A(z) = \infty$, otherwise; is the indicator of A.

The *Fitzpatrick function* of A is $h_A : Z \to \mathbb{R} \cup \{\infty\}$ defined by

$$h_A = p_A^*.$$

An alternative form for h_A is given by

$$h_A(z) = \sup_{a \in A}\{z \cdot a - p(a)\} = p(z) - \inf_{a \in A} p(z - a),\ \ z \in Z. \tag{9}$$

The conjugate of h_A

$$\varphi_A = h_A^* = \mathrm{cl}\ \mathrm{co}\ p_A, \tag{10}$$

is called the *Penot function* of A and it represents the greatest proper convex lower semicontinuous function majorized by p_A.

In the sequel, for $h : Z \to \mathbb{R} \cup \{\infty\}$, the following set notation will be frequently used

$$\{h = p\} = \{z \in Z;\ h(z) = p(z)\},$$

$$\{h \geq p\} = \{z \in Z;\ h(z) \geq p(z)\},$$

$$\{h \leq p\} = \{z \in Z;\ h(z) \leq p(z)\}.$$

PROPOSITION 2.1. *For every monotone $A \subset Z$ we have*

i) $\varphi_A(z) \geq p(z)$, for every $z \in Z$,

ii) $A \subset \{\varphi_A = p\}$,

iii) $A \subset \{h_A = p\}$,
iv) $D(A) \times X^* \subset \{h_A \geq p\}$.

Proof. iv) If $z = (x, x^*) \in D(A) \times X^*$ then there exists $\alpha^* \in Ax$. Let $a = (x, \alpha^*) \in A$. We have $z \cdot a - p(a) = p(z)$. According to (9), this yields $h_A(z) \geq p(z)$, i.e., $D(A) \times X^* \subset \{h_A \geq p\}$.

iii) Let $z \in A$. Pick $a = z$ in (9) to find $h_A(z) \geq p(z)$. Since A is monotone we get $p(z - a) = p(a) - z \cdot a + p(z) \geq 0$, for every $a \in A$. This gives us $h_A(z) \leq p(z)$. Therefore $A \subset \{h_A = p\}$ and $h_A \leq p_A$ in Z. Because h_A is proper convex lower semicontinuous this yields

$$h_A \leq \varphi_A \text{ in } Z, \tag{11}$$

for every A monotone.

Let $\mathcal{A}$ be a maximal monotone extension of A. By Theorem 3.3. below, we know that $h_{\mathcal{A}} \geq p$ in Z. We have $p_A \geq p_{\mathcal{A}}$ and

$$\varphi_A \geq \varphi_{\mathcal{A}} \geq h_{\mathcal{A}} \geq p \text{ in } Z, \tag{12}$$

i.e., i) holds.

Subpoint ii) is straight forward from i) and $\varphi_A \leq p_A$ in Z. The proof is complete. □

For other properties of h_A, φ_A see [12, Proposition 2].

3. Representability and maximality

DEFINITION 3.1. A multi-valued operator A is called *representable* in $Z = X \times X^*$ if there is a proper convex lower semicontinuous $h : Z \to \mathbb{R} \cup \{\infty\}$ such that

i) $h(z) \geq p(z)$, for every $z \in Z$, i.e., $\{h \geq p\} = Z$,
ii) $z \in A$ iff $h(z) = p(z)$, i.e., $A = \{h = p\}$.

A function h with properties i), ii) is called a *representative* of A. Notice that if h is a representative of A then from $A \subset \{h = p\}$ we get $h \leq p_A$ in Z followed by

$$h \leq \varphi_A, \; h^* \geq h_A \text{ in } Z. \tag{13}$$

LEMMA 3.1. ([5, Proposition 4]) *Every representable operator A is monotone.*

Proof. Since A is representable, there exists $h : Z \to \mathbb{R} \cup \{\infty\}$ such that $h \geq p$ in Z and $z \in A$ iff $h(z) = p(z)$. Therefore, from the convexity of h we get that, for every $z_1, z_2 \in A = \{h = p\}$

$$\frac{1}{2}p(z_1) + \frac{1}{2}p(z_2) - \frac{1}{4}p(z_1 - z_2) = p(\frac{1}{2}z_1 + \frac{1}{2}z_2)$$

$$\leq h(\frac{1}{2}z_1 + \frac{1}{2}z_2) \leq \frac{1}{2}h(z_1) + \frac{1}{2}h(z_2) = \frac{1}{2}p(z_1) + \frac{1}{2}p(z_2), \tag{14}$$

that is $p(z_1 - z_2) \geq 0$, for every $z_1, z_2 \in A$. $\square$

We prove that an operator is representable iff its Penot function becomes a representative.

THEOREM 3.2. *A is representable iff φ_A is a representative of A.*

Proof. For the direct implication let h be a representative of A. From $A = \{h = p\}$ we know that $h \leq p_A$ in Z. Therefore

$$\varphi_A \geq h \geq p, \text{ in } Z. \tag{15}$$

Combined with $A \subset \{\varphi_A = p\}$ the previous inequality shows that $A = \{\varphi_A = p\}$, which implies that φ_A is a representative of A. The converse implication is plain. $\square$

For different proofs of the previous result see e.g. [4,5].

The following characterization of maximality in terms of representability is due to Fitzpatrick [2, Theorem 3.8]. For the sake of completeness we provide the reader with a short proof.

THEOREM 3.3. *A multi-valued operator A is maximal monotone iff h_A is a representative of A.*

Proof. If A is maximal monotone then for every $z \notin A$ there exists an $a \in A$ such that $p(z - a) < 0$. Hence, from (9) we have $h_A(z) > p(z)$ for every $z \notin A$. Since $h_A(z) = p(z)$, for every $z \in A$ (see Proposition 2.1. iii)) this implies that $h_A \geq p$ in Z and $h_A(z) = p(z)$ iff $z \in A$, that is h_A is a representative of A.

Conversely, from Lemma 3.1. or from $h_A(z) = p(z)$ for every $z \in A$ and (9) we get $p(z - a) \geq 0$ for every $z, a \in A$, i.e., A is monotone.

Take $z_0 \in Z$ such that $p(z_0 - a) \geq 0$ for every $a \in A$. Again, from (9) we find $h_A(z_0) = p(z_0)$, that is $z_0 \in A$, since h_A is a representative of A. We showed that A is maximal monotone. The proof is complete. $\square$

Clearly, every maximal monotone operator is representable. The question whether the converse holds appears naturally in this context. The following characterization of maximality in terms of representability appeared first in Voisei [11, Theorem 2.3]. For the sake of convenience we provide the reader with a simpler proof.

THEOREM 3.4. *A is maximal monotone iff A is representable and $h_A \geq p$* in Z.

Proof. The direct implication is trivial since h_A is a representative of A.

Conversely, we know that A is monotone since it is representable. According to Proposition 2.1., we have $A \subset \{h_A = p\}$. To conclude that h_A is a representative of A and consequently that A is maximal monotone, it is enough to prove that $\{h_A = p\} \subset A$. Let $z \in \{h_A = p\}$. Clearly, z is a global minimum point of $h_A - p$. Therefore

$$0 \in \partial(h_A - p)(z), \tag{16}$$

where "∂" denotes the Clarke-Rockafellar subdifferential. Since p is continuously Gâteaux differentiable with $\partial(-p(z)) = \{-z\}$ and h_A is convex, relation (16) reduces to $z \in \partial h_A(z)$ which can be equivalently restated as

$$h_A(z) + \varphi_A(z) = 2p(z).$$

This implies $z \in \{\varphi_A = p\} = A$ because $h_A(z) = p(z)$ and A is representable. The proof is complete. $\square$

Remark 3.5. Condition $h_A \geq p$ in Z is sometimes referred to as A is of negative infimum type or NI in $X \times X^*$. Hence the previous characterization theorem can be restated as

$$\boxed{\text{Maximal Monotone} = \text{Representable+NI}}$$

This characterization of maximality is more versatile because most of the times the representability of operators is easily checked. Usually, the difficulty lies into proving that the operators are of NI type.

4. A simple proof of Heisler's result

Previous to the papers [11,12] there are two note-worthy results for the maximality of the sum in a non-reflexive Banach space setting; the result of Heisler for full-space domain operators and the result of Phelps & Simons (see [6, Theorem 7.2]) for linear single-valued operators. We provide a simpler proof of the Heisler result in order to observe the usefulness of the Fitzpatrick function and mention that in the linear multi-valued case the problem has been completely solved (see [12,13] or Theorem 5.13. below).

Recall the Heisler result

THEOREM 4.1. ([10, Theorem 37.4]) *Let X be a Banach space possibly non-reflexive. If A, B are maximal monotone in $X \times X^*$ with $D(A) = D(B) = X$ then $A + B$ is maximal monotone.*

The previous proof of this result relies on a topological characterization of maximal monotone operators with full-space domain. Our argument is based on the following two lemmas

LEMMA 4.2. *Let A be monotone with $D(A) = X$. Then A is of NI type in $Z = X \times X^*$, i.e.,*

$$h_A(z) \geq p(z), \text{ for every } z \in Z, \tag{17}$$

and $\{h_A = p\}$ is the only maximal monotone extension of A in $X \times X^*$.

Proof. According to Proposition 2.1. iv), $\{h_A = p\} = Z$, that is, A is (NI). For the second part notice that $\{h_A = p\}$ is representable monotone and every maximal monotone extension $\mathcal{A}$ of A satisfies

$$\mathcal{A} \subset \{h_A \leq p\} = \{h_A = p\}. \tag{18}$$

Therefore $\mathcal{A} = \{h_A = p\}$ and $\{h_A = p\}$ is the unique maximal monotone extension of A. □

LEMMA 4.3. *Let A be monotone with $D(A) = X$. Then A is maximal monotone iff A has convex values and A is closed with respect to the strong$\times$weakly-star convergence of bounded nets in $X \times X^*$ given by $(x_\alpha, x_\alpha^*) \twoheadrightarrow (x, x^*) \Leftrightarrow x_\alpha \to x$, strongly in X, $x_\alpha^* \to x^*$, weakly star in X^*, and $(x_\alpha^*)_\alpha$ is bounded in X^*.*

Proof. The direct implication is clear since every maximal monotone operator has convex values and is closed with respect to "$\twoheadrightarrow$".

For the converse it is enough to show that $\{h_A = p\} \subset A$.

Since A is closed with respect to "$\twoheadrightarrow$" we prove first that A has weakly-star closed values. Indeed, if $(x_\alpha^*)_\alpha \subset Ax$, $x \in X$, and $x_\alpha^* \to x^*$ weakly-star in X^* then $(x_\alpha = x, x_\alpha^*) \twoheadrightarrow (x, x^*)$ because Ax is bounded. Therefore, $x^* \in Ax$, i.e., Ax is weakly-star closed for every $x \in X$.

Let $z = (x_0, x_0^*) \in \{h_A = p\}$, that is, for every $(a, a^*) \in A$

$$\langle x_0 - a, x_0^* - a^* \rangle \geq 0. \tag{19}$$

Assume by contradiction that $x_0^* \notin Ax_0$. By a separation theorem we find $v_0 \in X$, such that

$$\langle v_0, x_0^* \rangle > \sup_{x^* \in Ax_0} \langle v_0, x^* \rangle. \tag{20}$$

For $t > 0$, denote by $a_t = x_0 + tv_0$ and take $a_t^* \in Aa_t$ in (19) to find

$$\langle v_0, x_0^* - a_t^* \rangle \leq 0. \tag{21}$$

Notice that for $t \downarrow 0$, $a_t \to x_0$, strongly in X. Because A is locally bounded at x_0, $(a_t^*)_t$ is bounded in X^*. Therefore, by the Alaoglu Theorem, at least on a subnet, we have $(a_t, a_t^*) \twoheadrightarrow (x_0, a_0^*) \in A$.

Pass to limit in (21) with $t \downarrow 0$ to get $\langle v_0, a_0^* \rangle \geq \langle v_0, x_0^* \rangle$ which contradicts (20). We proved $z \in A$, that is $A = \{h_A = p\}$. The proof is complete. □

Proof of Theorem 4.1. It is straight forward to show that if A, B are closed with respect to "$\twoheadrightarrow$" and have convex values then $A + B$ is "$\twoheadrightarrow$" closed and has

convex values, because A, B are locally bounded. According to Lemma 4.3., $A+B$ is maximal monotone. $\square$

5. Calculus rules for representable and maximal monotone operators

Our first concern in this section is the representability of $T := L^*ML : X \rightrightarrows X^*$, where X, Y are Banach spaces, $L : X \to Y$ is linear bounded, and $M : Y \rightrightarrows Y^*$ is representable. Let r_M be a representative of M.

Our choice for a representative of T is $r : X \times X^* \to \overline{\mathbb{R}}$,

$$r(x, x^*) = \inf\{r_M(Lx, y^*);\ L^*y^* = x^*\},\ (x, x^*) \in X \times X^*. \tag{22}$$

Notice that $r \geq p$ in Z, $T \subset \{r = p\}$, and $T = \{r = p\}$ whenever the "inf" in the definition of r is attained for all $(x, x^*) \in D(r)$. Therefore, it is enough to study conditions which assures that the "inf" in (22) becomes a "min".

For a subset S of a Banach space X we denote by ${}^iS$ the relative algebraic interior of S. We define ${}^{\mathrm{ic}}S = {}^iS$ if the affine hull of S is closed and ${}^{\mathrm{ic}}S = \emptyset$ otherwise.

THEOREM 5.1. *Let X, Y be Banach spaces, $L : X \to Y$ be linear bounded, $M : Y \rightrightarrows Y^*$ be representable, and r_M be a representative of M. If*

$$0 \in {}^{\mathrm{ic}}(R(L) - P_Y D(r_M^*)), \tag{23}$$

then $T := L^*ML : X \rightrightarrows X^*$ is representable. Here $P_Y : Y \times Y^* \to Y$, $P_Y(y, y^*) = y$, $(y, y^*) \in Y \times Y^*$ is the projection of $Y \times Y^*$ onto Y, $L^* : Y^* \to X^*$ denotes the adjoint of L, $R(L)$ is the range of L, and r_M^* stands for the convex conjugate with respect to the natural duality.

Proof. Consider $\varphi : X \times X^* \to \overline{\mathbb{R}}$,

$$\varphi(x, x^*) = \inf\{r_M^*(Lx, y^*);\ L^*y^* = x^*\},\ (x, x^*) \in X \times X^*. \tag{24}$$

Since $\varphi(x, x^*) = \inf\{r_M^*(y, y^*);\ (y, y^*) \in C(x, x^*)\}$, where the process $C \subset X \times X^* \times Y \times Y^*$ is defined by

$$(x, x^*, y, y^*) \in C \text{ iff } y = Lx,\ x^* = L^*y^*, \tag{25}$$

$R(C) = R(L) \times Y^*$, and condition $0 \in {}^{\mathrm{ic}}(R(L) - P_Y D(r_M^*))$ is equivalent to $0 \in {}^{\mathrm{ic}}(R(C) - D(r_M^*))$, according to [15, T 2.8.6], we may apply the chain rule to get

$$\varphi_{Z^*}^*(x^*, x^{**}) = \min\{r_M^{**}(y^*, y^{**});\ (x^*, x^{**}) \in C^*(y^*, y^{**})\},\ (x^*, x^{**}) \in X^* \times X^{**}. \tag{26}$$

Here $\varphi_{Z^*}^*$ denotes the convex conjugate of φ in $Z^* = X^* \times X^{**}$ and is weakly-star lower semicontinuous in Z^* which makes $\varphi^* = \varphi_{Z^*}^*/_Z$ weakly$\times$weakly-star lower semi-continuous in Z.

The adjoint of C is given by

$$(y^*, y^{**}, x^*, x^{**}) \in C^* \text{ iff } y^{**} = L^{**}x^{**},\ x^* = L^*y^*. \tag{27}$$

By the bi-conjugate formula for $x^{**} = x \in X$ relation (26) becomes

$$r(x, x^*) = \varphi^*(x, x^*) = \min\{r_M(Lx, y^*);\ L^*y^* = x^*\},\ (x, x^*) \in X \times X^*. \tag{28}$$

Relation (28) shows that r is a representative of T and consequently T is representable. $\square$

PROPOSITION 5.2. *Let X be a Banach space, $L : X \to X \times X$, $Lx = (x, x)$, $x \in X$, and U, V be convex subsets of X. Then*

$$0 \in {}^i(R(L) - U \times V) \Leftrightarrow 0 \in {}^i(U - V). \tag{29}$$

$$0 \in {}^{\mathrm{ic}}(R(L) - U \times V) \Leftrightarrow 0 \in {}^{\mathrm{ic}}(U - V). \tag{30}$$

Proof. Consider the "difference function" $\mathcal{D} : X \times X \to X$,

$$\mathcal{D}(x_1, x_2) = x_2 - x_1,\ x_1, x_2 \in X.$$

Notice that $\mathrm{Ker}\mathcal{D} = R(L)$, and let "aff" denote the affine hull of a subset in X or $X \times X$. We have

$$F := \mathrm{aff}(R(L) - U \times V) = R(L) - \mathrm{aff}U \times \mathrm{aff}V, \tag{31}$$

$$\mathcal{D}(F) = \mathrm{aff}(U - V). \tag{32}$$

According to [15, Corollary 1.3.15] $\mathcal{D}(F)$ is closed iff $F = F + \mathrm{Ker}\mathcal{D}$ is closed. Therefore, $\mathrm{aff}(R(L) - U \times V)$ is closed iff $\mathrm{aff}(U - V)$ is closed.

Since U, V are convex, condition $0 \in {}^i(R(L) - U \times V)$ is equivalent to $\bigcup_{n \geq 1} n(R(L) - U \times V)$ is a linear subspace and $0 \in {}^i(U - V)$ is equivalent to $\bigcup_{n \geq 1} n(U - V)$ is a linear subspace (see [15, (1.1)]).

But

$$(x_1, x_2) \in R(L) - U \times V \text{ iff } \mathcal{D}(x_1, x_2) = x_2 - x_1 \in U - V, \tag{33}$$

which shows that $\bigcup_{n \geq 1} n(R(L) - U \times V)$ is a linear subspace iff $\bigcup_{n \geq 1} n(U - V)$ is a linear subspace. Hence $0 \in {}^i(R(L) - U \times V)$ iff $0 \in {}^i(U - V)$. The proof is complete. $\square$

For a generalization of (29) see [16, Proposition 2.1].

THEOREM 5.3. *Let X be a Banach space and $A, B : X \rightrightarrows X^*$ be representable with*

$$0 \in {}^{\mathrm{ic}}(P_X D(r_A^*) - P_X D(r_B^*)), \tag{34}$$

where r_A, r_B are representatives of A, B and P_X stands for the projection of $X \times X^*$ onto X. Then $A + B$ is representable.

Proof. First argument. We apply Theorem 5.1. for X, $Y = X \times X$, $Lx = (x, x)$, $x \in X$, and $M(x_1, x_2) = Ax_1 \times Bx_2$, $(x_1, x_2) \in D(M) = D(A) \times D(B)$ for which $L^*ML = A + B$,

$$r_M(x_1, x_2, x_1^*, x_2^*) = r_A(x_1, x_1^*) + r_B(x_2, x_2^*),\ x_1, x_2 \in X,\ x_1^*, x_2^* \in X^*, \tag{35}$$

is a representative of M with

$$r_M^*(x_1, x_2, x_1^*, x_2^*) = r_A^*(x_1, x_1^*) + r_B^*(x_2, x_2^*),\ x_1, x_2 \in X,\ x_1^*, x_2^* \in X^*, \tag{36}$$

$P_Y D(r_M^*) = P_X D(r_A^*) \times P_X D(r_B^*)$, and according to Proposition 5.2., condition $0 \in {}^{\mathrm{ic}}(P_X D(r_A^*) - P_X D(r_B^*))$ is equivalent to $0 \in {}^{\mathrm{ic}}(R(L) - P_Y D(r_M^*))$.

Second argument. Let $(x_0, x_0^*) \in \{\varphi_{A+B} = p\}$. Since $\varphi_{A+B} = h_{A+B}^*$ we have that for every $(u, u^*) \in Z$

$$h_{A+B}(u, u^*) - \langle u, x_0^* \rangle - \langle x_0, u^* \rangle + \langle x_0, x_0^* \rangle \geq 0. \tag{37}$$

Let $\mathcal{X} = X \times X^* \times X^*$, $\mathcal{Y} = X$ and consider the function $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{\infty\}$ given by

$$\Phi(x, x^*, z^*; y) = r_A^*(x + y, x^*) + r_B^*(x, z^*) - \langle x, x_0^* \rangle - \langle x_0, x^* + z^* \rangle + \langle x_0, x_0^* \rangle,$$

$x, y \in X$, $x^*, z^* \in X^*$.

Notice that, since $r_A^* \geq h_A$, $r_B^* \geq h_B$ (see (13)), we get

$$\begin{aligned} \Phi(x, x^*, z^*; 0) &\geq h_A(x, x^*) + h_B(x, z^*) - \langle x, x_0^* \rangle - \langle x_0, x^* + z^* \rangle + \langle x_0, x_0^* \rangle \\ &\geq h_{A+B}(x, x^* + z^*) - \langle x, x_0^* \rangle - \langle x_0, x^* + z^* \rangle + \langle x_0, x_0^* \rangle \geq 0, \end{aligned}$$

for every $(x, x^*, z^*) \in \mathcal{X}$, i.e., $\inf_{\chi \in \mathcal{X}} \Phi(\chi, 0) \geq 0$.

If $P_{\mathcal{Y}}(\chi, y) = y$, $(\chi, y) \in \mathcal{X} \times \mathcal{Y}$, is the projection of $\mathcal{X} \times \mathcal{Y}$ onto $\mathcal{Y}$ then $P_{\mathcal{Y}} D(\Phi) = P_X D(r_A^*) - P_X D(r_B^*)$ and condition (34) spells $0 \in {}^{\mathrm{ic}}(P_{\mathcal{Y}} D(\Phi))$.

This allows us to apply the fundamental duality formula (see e.g. [15, Theorem 2.7.1 (vii)]) to get

$$\inf_{\chi \in \mathcal{X}} \Phi(\chi, 0) = \max_{y^* \in Y^* = X^*} (-\Phi^*(0, y^*)) \geq 0. \tag{38}$$

Therefore, there exists $y^* \in X^*$ such that

$$\Phi^*(0, y^*) = \sup\{\langle y, y^* \rangle - \Phi(x, x^*, z^*; y);\ x, y \in X,\ x^*, z^* \in X^*\} \leq 0, \tag{39}$$

that is

$$r_A^*(x+y,x^*)+r_B^*(x,z^*)-\langle x,x_0^*\rangle-\langle x_0,x^*+z^*\rangle+\langle x_0,x_0^*\rangle-\langle y,y^*\rangle\geq 0, \quad (40)$$

for every $x,y\in X$, $x^*,z^*\in X^*$.

Using the substitution $x+y=z$, we find

$$r_A^*(z,x^*)+r_B^*(x,z^*)-\langle x,x_0^*\rangle-\langle x_0,x^*+z^*\rangle+\langle x_0,x_0^*\rangle-\langle z-x,y^*\rangle\geq 0, \quad (41)$$

for every $x,y\in X$, $x^*,z^*\in X^*$, or

$$[\langle z,y^*\rangle+\langle x_0,x^*\rangle-r_A^*(z,x^*)]+[\langle x,x_0^*-y^*\rangle+\langle x_0,z^*\rangle-r_B^*(x,z^*)]\leq\langle x_0,x_0^*\rangle,$$

for every $x,y\in X$, $x^*,z^*\in X^*$, that is

$$r_A(x_0,y^*)+r_B(x_0,x_0^*-y^*)\leq\langle x_0,x_0^*\rangle. \quad (42)$$

Because r_A,r_B are representatives of A,B relation (42) is equivalent to

$$r_A(x_0,y^*)=\langle x_0,y^*\rangle,\ \ r_B(x_0,x_0^*-y^*)=\langle x_0,x_0^*-y^*\rangle, \quad (43)$$

that is, $(x_0,y^*)\in A$, $(x_0,x_0^*-y^*)\in B$. Hence $(x_0,x_0^*)\in A+B$. We proved that $\{\varphi_{A+B}=p\}\subset A+B$ and this is enough in order to conclude that $A+B$ is representable. □

Remark 5.4. The typical example of a representative of A is provided by the Penot function φ_A. Therefore, in a particular case, Theorems 5.1, 5.3 can be restated as

COROLLARY 5.5. *Let X,Y be two Banach spaces, $L:X\to Y$ be linear bounded, and $M:Y\rightrightarrows Y^*$ be representable. If*

$$0\in{}^{\rm ic}(R(L)-P_YD(h_M)) \quad (44)$$

then $T:=L^*ML:X\rightrightarrows X^*$ is representable.

COROLLARY 5.6. *Let X be a Banach space and $A,B:X\rightrightarrows X^*$ be representable with*

$$0\in{}^{\rm ic}(P_XD(h_A)-P_XD(h_B)).$$

Then $A+B$ is representable.

PROPOSITION 5.7. *If $M:Y\rightrightarrows Y^*$ is monotone in the Banach space Y, $D(M)$ is closed convex, and*

$$M=M+N_{D(M)} \quad (45)$$

then M is of NI type and $D(M)=P_YD(h_M)$. Here $N_{D(M)}$ stands for the convex normal cone to $D(M)$.

Proof. For every $y \in D(M)$ there is $y^* \in Y^*$ such that $(y, y^*) \in M \subset \{h_M = p\} \subset D(h_M)$, that is, $D(M) \subset P_Y D(h_M)$.

Conversely, let $y \in P_Y D(h_M)$, that is, $h_M(y, y^*) < \infty$, for some $y^* \in Y^*$. Hence, for every $(m, m^*) \in M$, we have

$$\langle y - m, m^* \rangle + \langle m, y^* \rangle \leq C < \infty. \tag{46}$$

From (45), (46) and because $N_{D(M)}(y)$ is a cone for every $y \in D(M)$, we get

$$t\langle y - m, n^* \rangle + \langle y - m, m^* \rangle + \langle m, y^* \rangle \leq C < \infty, \tag{47}$$

for every $t > 0$, $m \in D(M)$, $m^* \in Mm$, $n^* \in N_{D(M)}(m)$.

From (47) it yields that $\langle y - m, n^* \rangle \leq 0$, for every $m \in D(M)$, $n^* \in N_{D(M)}(m)$, i.e., $(y, 0)$ is monotonically related to the graph of the maximal monotone operator $N_{D(M)}$. Therefore, $(y, 0) \in N_{D(M)}$, that is, $y \in D(M)$. We proved $P_Y D(h_M) = D(M)$, i.e., $D(h_M) \subset D(M) \times X^*$. According to Proposition 2.1. iv) this implies $D(h_M) \subset \{h_M \geq p\}$, that is M is NI. $\square$

Remark 5.8. Condition (45) is satisfied whenever M is maximal monotone, since $N_{D(M)}$ is monotone, $0 \in N_{D(M)}(y)$ for every $y \in D(M)$, and $M \subset M + N_{D(M)}$. Therefore, every maximal monotone M with $D(M)$ closed convex has $P_Y D(h_M) = D(M)$.

THEOREM 5.9. *Let X, Y be Banach spaces, $L : X \to Y$ be linear bounded, $M : Y \rightrightarrows Y^*$ be maximal monotone, and $T := L^* M L : X \rightrightarrows X^*$.*

(α) If $D(T)$ is closed convex and

$$N_{D(T)} = L^* N_{D(M)} L, \tag{48}$$

then T is of NI type.

(β) If $D(M)$ is closed convex and

$$0 \in {}^{\mathrm{ic}}(R(L) - D(M)), \tag{49}$$

then T is maximal monotone.

(γ) If $D(T)$ is closed and $R(L) \cap \operatorname{int} D(M) \neq \emptyset$ then T is maximal monotone.

Proof. (α) Since M is maximal monotone we know that $M = M + N_{D(M)}$. We find $M(Lx) = M(Lx) + N_{D(M)}(Lx)$ and

$$Tx = L^* M(Lx) = L^* M(Lx) + L^* N_{D(M)}(Lx) = Tx + N_{D(T)}(x),$$

for every $x \in D(T) = L^{-1}(D(M))$, that is, $T = T + N_{D(T)}$ and, according to Proposition 5.7., T is of NI type.

(β) Since $D(M)$ is closed convex, $D(T) = L^{-1}(D(M))$ is closed convex, $D(M) = P_Y D(h_M)$, and (49) becomes (44), and so, by Corollary 5.5., T is representable. Also,

$$i_{D(T)}(x) = \inf\{i_{D(M)}(y);\ y = Lx\},\ x \in D(T). \tag{50}$$

Taking into account (49), we may apply the chain rule [15, Theorem 2.8.6 (v)] to get that

$$N_{D(T)} = L^* N_{D(M)} L. \tag{51}$$

According to (α), T is of NI type. Hence T is maximal monotone.

(γ) Because M is maximal monotone with $\mathrm{int}D(M) \neq \emptyset$, $\mathrm{int}D(M)$, $\overline{D(M)}$ are convex, $\mathrm{int}D(M) = \mathrm{int}\overline{D(M)}$, $\overline{D(M)} = \overline{\mathrm{int}D(M)}$ (see e.g [10, Theorem 18.4]), $\mathrm{int}D(M) = \mathrm{int}P_Y D(h_M)$ (see e.g. [9, Theorem 2.2.]), $R(L) - P_Y D(h_M)$ contains 0 in its interior, and (44) follows making T representable.

We prove that

$$D(T) = L^{-1}(\overline{D(M)}). \tag{52}$$

The direct inclusion is plain since L is continuous and $D(T)$ is closed.

Conversely, let $x_0 \in L^{-1}(\overline{D(M)})$, that is, $Lx_0 \in \overline{D(M)}$. Without loss of generality we may assume that $0 \in \mathrm{int}D(M)$ and $0 \in M0$. Then $\lambda L x_0 \in D(M)$, for every $0 \leq \lambda < 1$ (see e.g. [15, Theorem 1.1.2]), i.e., $\lambda x_0 \in D(T)$, for $0 \leq \lambda < 1$. Letting $\lambda \uparrow 1$, we find $x_0 \in \overline{D(T)} = D(T)$.

Relation (52) shows that $D(T)$ is closed convex.

Again, from the chain rule [15, Theorem 2.8.6 (iii)] applied for

$$i_{D(T)}(x) = i_{L^{-1}(\overline{D(M)})}(x) = \inf\{i_{\overline{D(M)}}(y);\ y = Lx\},\ x \in X.$$

we get (48), that is T is NI and this is sufficient in order to conclude. □

THEOREM 5.10. *Let A, B be maximal monotone operators in the Banach space X.*

(α) If $D(A) \cap D(B)$ is closed convex and

$$N_{D(A)\cap D(B)} = N_{D(A)} + N_{D(B)}, \tag{53}$$

then $A + B$ is of NI type.

(β) If $D(A)$, $D(B)$ are closed convex and

$$0 \in {}^{\mathrm{ic}}(D(A) - D(B)), \tag{54}$$

then $A + B$ is maximal monotone,

(γ) If $D(A) \cap D(B)$ is closed, $\overline{D(A)}$ is convex, and $D(A) \cap \mathrm{int}D(B) \neq \emptyset$ then $\overline{D(A)} \cap \overline{D(B)} = D(A) \cap D(B)$ and $A + B$ is maximal monotone.

(δ) If $D(A)$ is closed convex and $D(A) \subset D(B)$ then $A + B$ is of NI type.

(ϵ) If $D(A)$ is closed convex, $D(A) \subset D(B)$, and $0 \in {}^{\mathrm{ic}}(D(A) - P_X D(h_B))$ then $A + B$ is maximal monotone.

Proof. Sub-points (α), (β) are direct consequences of Theorem 5.9. (α), (β) applied for $Y = X \times X$, $Lx = (x, x)$, $x \in X$, $L^* : Y^* = X^* \times X^* \to X^*$, $L^*(x^*, y^*) = x^* + y^*$, $x^*, y^* \in X^*$, $M(x_1, x_2) = Ax_1 \times Bx_2$, $(x_1, x_2) \in D(M) = D(A) \times D(B)$, for which $L^*ML = A + B$. More precisely, subpoint (β) follows from Theorem 5.9. (β) since $0 \in {}^{\mathrm{ic}}(D(A) - D(B))$ iff $0 \in {}^{\mathrm{ic}}(R(L) - D(M))$ (see (30)). For an alternative proof of (β) see [12, Theorem 2].

(γ) Without loss of generality assume that $0 \in D(A) \cap \mathrm{int} D(B)$. If $x \in \overline{D(A)} \cap \overline{D(B)}$, then, for every $0 \le \lambda < 1$, $\lambda x \in \overline{D(A)} \cap \mathrm{int} D(B) \subset \overline{D(A) \cap D(B)}$. Let $\lambda \uparrow 1$ to find $x \in \overline{D(A) \cap D(B)} = D(A) \cap D(B)$, that is $\overline{D(A)} \cap \overline{D(B)} = D(A) \cap D(B)$ and consequently $D(A) \cap D(B)$ is convex.

Therefore, for every $x \in D(A) \cap D(B)$

$$N_{D(A)\cap D(B)}(x) = N_{\overline{D(A)}\cap\overline{D(B)}}(x)$$

$$= N_{\overline{D(A)}}(x) + N_{\overline{D(B)}}(x) = N_{D(A)}(x) + N_{D(B)}(x),$$

i.e., (53) holds. The NI type follows from (α) while the representability is a consequence of $D(A) \cap \mathrm{int} D(B) \neq \emptyset$ and Corollary 5.6.

(δ) Clearly, $D(A) \cap D(B) = D(A)$ is closed convex and since $N_{D(A)}$ is maximal monotone we get

$$N_{D(A)\cap D(B)} = N_{D(A)} = N_{D(A)} + N_{\overline{co}D(B)} = N_{D(A)} + N_{D(B)},$$

i.e., according to (α), $A + B$ is NI. Here "$\overline{co}$" stands for the closed convex hull.

(ϵ) Condition $0 \in {}^{\mathrm{ic}}(D(A) - P_X D(h_B))$ implies the representability of $A + B$. From (δ) we know that $A + B$ is NI, therefore $A + B$ is maximal monotone. □

Remark 5.11. A recent results of Groh [3, Theorem 1.6] is a particular case of our subpoint (ϵ), for A being a subdifferential and B having a non-empty interior.

The following result of Bauschke presents a different perspective on the subject.

THEOREM 5.12. ([10, Theorem 39.1]) *Let A be maximal monotone in the Banach space X and $B : X \to X^*$ be linear with $\langle Bx, x\rangle = 0$, for every $x \in X$. Then $A + B$ is maximal monotone.*

Proof. It is easily checked that for every $(x, x^*) \in X \times X^*$

$$h_{A+B}(x, x^*) = h_A(x, x^* + B^*x) = h_A(x, x^* - Bx), \tag{55}$$

where $B^* = -B$ stands for the adjoint of B. This equality suffices in order to conclude that h_{A+B} is a representative of $A + B$ and $A + B$ is maximal monotone. □

Notice that under the assumptions of Bauschke's result we have

$$h_{A+B}(x,x^*) = \inf\{h_A(x,y^*) + h_B(x,z^*);\ y^* + z^* = x^*\} = (h_A \square_2 h_B)(x,x^*), \tag{56}$$

for every $(x,x^*) \in X \times X^*$, since $h_B(x,z^*) = 0$, iff $z^* = -B^*x$, $h_B(x,z^*) = +\infty$, otherwise; where "$\square_2$" denotes the infimal convolution with respect to the second variable.

It is worth noticing that equality (56) assures that $A+B$ is NI and that $A+B$ is maximal monotone whenever the infimal convolution in (56) is exact. Unfortunately, in general (56) does not hold even under the assumptions $D(A) = D(B) = X$ and X is a Hilbert space (see e.g. [7, Example 1]). Other cases in which an equality of type (56) holds are given in the following theorem.

THEOREM 5.13. *Let X, Y be Banach spaces.*

(α) If $L : X \to Y$ is linear bounded, and $M : Y \rightrightarrows Y^*$ is maximal monotone with Graph(M) convex in $X \times X^*$ and

$$0 \in {}^{\mathrm{ic}}(R(L) - D(M)), \tag{57}$$

then $T := L^*ML : X \rightrightarrows X^*$ is maximal monotone.

(β) If A, B are maximal monotone operators in X with Graph(A), Graph(B) convex and

$$0 \in {}^{\mathrm{ic}}(D(A) - D(B)), \tag{58}$$

then $A+B$ is maximal monotone.

Proof. (α) We have

$$\begin{aligned} p_T(x,x^*) &= \min\{p_M(Lx,y^*);\ L^*y^* = x^*\} \\ &= \min\{p_M(y,y^*);\ (y,y^*) \in C(x,x^*)\},\ (x,x^*) \in X \times X^*, \end{aligned} \tag{59}$$

where $C \subset X \times X^* \times Y \times Y^*$ is defined in (25) with adjoint C^* given by (27).

Notice that $R(C) = R(L) \times Y^*$, $D(p_M) = M$, $R(C) - D(p_M) = (R(L) - D(M)) \times Y^*$ and condition $0 \in {}^{\mathrm{ic}}(R(C) - D(p_M))$ is equivalent to (57). Moreover, Graph$M$ strongly closed and convex in $X \times X^*$ makes p_M proper convex strongly lower semicontinuous in $X \times X^*$.

We apply the chain rule [15, Theorem 2.8.6 (v)] to get

$$p_T^*(x^*,x^{**}) = \min\{p_M^*(y^*,y^{**});\ (x^*,x^*) \in C^*(y^*,y^{**})\}, \tag{60}$$

$(x,x^*) \in X \times X^*$. For $x^{**} = x \in X$ we find

$$h_T(x,x^*) = \min\{h_M(Lx,y^*);\ L^*y^* = x^*\}, \tag{61}$$

which implies that h_T is a representative of T, i.e., T is maximal monotone.

(β) Again, take $Y = X \times X$, $Lx = (x,x)$, $x \in X$, $L^* : Y^* = X^* \times X^* \to X^*$, $L^*(x^*,y^*) = x^* + y^*$, $x^*, y^* \in X^*$, and $M(x_1,x_2) = Ax_1 \times Bx_2$, $(x_1,x_2) \in D(M) = D(A) \times D(B)$ or Graph$M$ = Graph$A \times$ Graph$B$.

Then $L^*ML = A + B$ is maximal monotone by the conclusion of (α), taking into consideration that, in this case, (58) is equivalent to (57). $\square$

COROLLARY 5.14. *Let X, Y be Banach spaces.*

(α) If $L : X \to Y$ is linear bounded, and $M : Y \rightrightarrows Y^*$ is linear maximal monotone with $R(L) - D(M)$ closed in Y, then $T := L^*ML : X \rightrightarrows X^*$ is maximal monotone.

(β) If A, B are linear maximal monotone with $D(A) - D(B)$ closed in X then $A + B$ is maximal monotone.

Proof. Condition (57) is equivalent to $R(L) - D(M)$ closed in Y, since $R(L) - D(M)$ is a subspace. Similarly, (58) becomes $D(A) - D(B)$ is closed in X. For a different proof of (β) see [13]. $\square$

It is worth mentioning that in the linear case the qualification constraints contained in (α), (β) cannot be further relaxed (see e.g. [10] for a counter-example).

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